Parallel Transient Algorithm with Multistep Substructure Computation

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A parallel algorithm for computing the transient response of structures is presented. The computation is parallelized on the basis of a division of the structure into substructures, with each processor computing the response of a substructure independently. Independently computed substructure responses are reconciled directly, rather than iteratively, to obtain the solution of the global problem. The only data required for correcting the independently computed response of a substructure is the interface motion computed independently for other substructures. Reconciliation of substructure responses is not required after every time step; instead, it can be postponed until after the responses have been computed independently for multiple time steps. A numerical example is presented that demonstrates the method and its accuracy.

Introduction

FREQUENTLY in current engineering practice, transient response must be computed for structures whose finite element models have 10,000 or more degrees of freedom. The computational burden associated with this is usually reduced by employing model reduction techniques such as modal truncation, condensation methods, and component mode synthesis. The use of these techniques can require considerable effort on the part of an analyst, and their use can result in unacceptable loss of accuracy if they are not used very carefully. Because it may be difficult or impossible to maintain acceptable accuracy while reducing the order of transient response problems to satisfy computational constraints, there is a need for new algorithms for handling very large transient response problems. In particular, because multiprocessor or parallel computers are becoming more accessible, and because of the improvement in computational capability that multiprocessors offer, there is a need for algorithms that exploit the full capabilities of multiprocessors in the solution of these problems.

The question of how to divide the computational effort among the available processors must be addressed. In this paper the computation is parallelized on the basis of a division of the structure model into substructures. This is an example of the domain decomposition approach to parallelization, in which different subdomains of the overall problem domain are handled by different processors.[1-3] An advantage of the domain decomposition approach over some other approaches to parallelization is that each processor primarily needs to access data associated with a single subdomain. This can simplify data transfer between processors and memory considerably for some parallel architectures.

A number of algorithms have been developed involving independent computation of transient response for different subdomains. Some of these have been motivated by the need to solve problems for systems consisting of two or more well-defined subsystems, such as the Shuttle Orbiter and its payloads, using modal data for subsystems that has already been obtained, rather than computing new modal data for the combined system.[4-6] Other algorithms have been developed in the context of element-by-element finite element analysis.[7,8] Ortiz et al.[9,10] have proposed methods for concurrent computation of subdomain transient response in which an implicit integration scheme is used to compute the response for each subdomain for a given time step, and the results of these computations are averaged at interfaces to yield an approximation of the response of the overall system. Hajjar and Abel[11] have investigated the accuracy of these latter methods for certain frame dynamics applications and have concluded that their accuracy is inadequate for these problems when practical time step sizes are used. Admire and Brunty[12] have developed a transient response method with some qualitative similarities to the method of this paper, but with an apparent time step restriction that the method presented here is not subject to. In all of the algorithms mentioned, computation for substructures can only proceed independently for one time step at a time.

In the algorithm of this paper the response is first computed independently for different substructures, and then independently computed structure responses are reconciled in such a way that the computed response satisfies the structure equations of motion at the end of every time step. The procedure for doing this is direct, rather than iterative. The only data that are required for correcting the response within one substructure are the interface portions of independently computed responses for other substructures. A significant feature of this method is that it permits the computation of the substructure response to proceed independently for multiple time steps before any reconciliation between substructure responses is done. This allows communication between processors to be done less frequently and decreases interdependence between processors.

This paper is organized as follows. In the next section a simple method is presented for computing substructure responses independently and then reconciling them iteratively to obtain the response of the overall structure. In the following section the iterative method for reconciliation is replaced with an efficient direct method. The next section presents procedures for carrying out the reconciliation occasionally, rather than at every time step. Then a numerical example is presented, and the final section contains conclusions.

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Iterative Method for Parallel Response Computation

After a complex structure has been discretized in space, typically by the finite element method, its response is usually assumed to be governed by equations of motion having the form

$$M\ddot{u} + C\dot{u} + Ku = F(t) \tag{1}$$

where M, C, and K are mass, damping, and stiffness matrices, respectively; $\ddot{u}$, $\dot{u}$, and u are acceleration, velocity, and displacement vectors, respectively; and $F(t)$ is an excitation vector for the system. The parallel algorithm presented in this paper has been developed for linear transient response problems of this type, where the matrices M, C, and K are time-invariant. The system matrices are often symmetric, but need not be so for the algorithm presented here. As mentioned in the Introduction, the computation is parallelized on the basis of a division of the structure into substructures. In terms of the notation that will be used in this paper, the mass matrix for a structure that can be partitioned into two substructures can be written in the following form, with a possible reordering of rows and columns:

$$M = \begin{bmatrix} M_{LL}^{(1)} & M_{LS}^{(1)} & 0 \\ M_{SL}^{(1)} & M_{SS}^{(1)} + M_{SS}^{(2)} & M_{SL}^{(2)} \\ 0 & M_{LS}^{(2)} & M_{LL}^{(2)} \end{bmatrix} \tag{2}$$

Superscripts in parentheses identify the substructure that a given matrix partition is associated with, and the subscripts L and S refer to matrix partitions associated with *local*, or internal, and *shared*, or interface, degrees of freedom. A two-substructure structure is used for much of the presentation in this paper to minimize unnecessary complication.

The objective of this paper is to develop an algorithm for computing the transient response of structures using simultaneous, independent substructure response computation. To this end, independent substructure response problems can be extracted from the structure problem that have the form

$$\begin{bmatrix} M_{LL}^{(k)} & M_{LS}^{(k)} \\ M_{SL}^{(k)} & \hat{M}_{SS}^{(k)} \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Lo}^{(k)} \\ \ddot{u}_{So}^{(k)} \end{Bmatrix} + \begin{bmatrix} C_{LL}^{(k)} & C_{LS}^{(k)} \\ C_{SL}^{(k)} & \hat{C}_{SS}^{(k)} \end{bmatrix} \begin{Bmatrix} \dot{u}_{Lo}^{(k)} \\ \dot{u}_{So}^{(k)} \end{Bmatrix} + \begin{bmatrix} K_{LL}^{(k)} & K_{LS}^{(k)} \\ K_{SL}^{(k)} & \hat{K}_{SS}^{(k)} \end{bmatrix} \begin{Bmatrix} u_{Lo}^{(k)} \\ u_{So}^{(k)} \end{Bmatrix} = \begin{Bmatrix} F_L^{(k)} \\ \hat{F}_S^{(k)} \end{Bmatrix} \tag{3}$$

where carets over an interface partition of a matrix or vector indicates that some modification of these may be appropriate because of the interaction between substructures at interfaces in the structure problem. The o subscripts identify the solutions of these equations as the original independently computed substructure responses, which will have to be corrected to obtain responses of substructures in the response of the structure. Solving independent response problems like these for each of the substructures will result in different interface responses associated with different substructures, and these must be combined in some manner to obtain a representation of the response of the structure at interfaces. Although a number of approaches are possible, a convention is arbitrarily adopted throughout this paper in which the structure response at an interface is represented as the sum of responses associated with the substructures sharing the interface. Using this convention, the displacement for a two-substructure structure is represented in the form

$$u = \begin{Bmatrix} u_L^{(1)} \\ u_S^{(1)} + u_S^{(2)} \\ u_L^{(2)} \end{Bmatrix} \tag{4}$$

which is reminiscent of assembly of global matrices from element matrices in the finite element method.

After the response is computed independently for different substructures and assembled together in this manner, the residual in the structure equations of motion becomes, for a two-substructure structure,

$$\begin{aligned} r(t) &= M\ddot{u} + C\dot{u} + Ku - F \\ &= \begin{bmatrix} M_{LL}^{(1)} & M_{LS}^{(1)} & 0 \\ M_{SL}^{(1)} & M_{SS}^{(1)} + M_{SS}^{(2)} & M_{SL}^{(2)} \\ 0 & M_{LS}^{(2)} & M_{LL}^{(2)} \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Lo}^{(1)} \\ \ddot{u}_{So}^{(1)} + \ddot{u}_{So}^{(2)} \\ \ddot{u}_{Lo}^{(2)} \end{Bmatrix} \\ &+ \begin{bmatrix} C_{LL}^{(1)} & C_{LS}^{(1)} & 0 \\ C_{SL}^{(1)} & C_{SS}^{(1)} + C_{SS}^{(2)} & C_{SL}^{(2)} \\ 0 & C_{LS}^{(2)} & C_{LL}^{(2)} \end{bmatrix} \begin{Bmatrix} \dot{u}_{Lo}^{(1)} \\ \dot{u}_{So}^{(1)} + \dot{u}_{So}^{(2)} \\ \dot{u}_{Lo}^{(2)} \end{Bmatrix} \\ &+ \begin{bmatrix} K_{LL}^{(1)} & K_{LS}^{(1)} & 0 \\ K_{SL}^{(1)} & K_{SS}^{(1)} + K_{SS}^{(2)} & K_{SL}^{(2)} \\ 0 & K_{LS}^{(2)} & K_{LL}^{(2)} \end{bmatrix} \begin{Bmatrix} u_{Lo}^{(1)} \\ u_{So}^{(1)} + u_{So}^{(2)} \\ u_{Lo}^{(2)} \end{Bmatrix} \\ &- \begin{Bmatrix} F_L^{(1)} \\ F_S \\ F_L^{(2)} \end{Bmatrix} \end{aligned} \tag{5}$$

The motion of the structure that is obtained as a result of the independent substructure response calculations, with summation at the interfaces, satisfies a structure response problem with an excitation equal to $F(t) + r(t)$. By making use of Eq. (3), the residual can be obtained as

$$\begin{Bmatrix} r_L^{(1)}(t) \\ r_S(t) \\ r_L^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} M_{LS}^{(1)}\ddot{u}_{So}^{(2)} + C_{LS}^{(1)}\dot{u}_{So}^{(2)} + K_{LS}^{(1)}u_{So}^{(2)} \\ r_S(t) \\ M_{LS}^{(2)}\ddot{u}_{So}^{(1)} + C_{LS}^{(2)}\dot{u}_{So}^{(1)} + K_{LS}^{(2)}u_{So}^{(1)} \end{Bmatrix} \tag{6}$$

where

$$\begin{aligned} r_S(t) &= (M_{SS}^{(1)} + M_{SS}^{(2)})(\ddot{u}_{So}^{(1)} + \ddot{u}_{So}^{(2)}) - \hat{M}_{SS}^{(1)}\ddot{u}_{So}^{(1)} - \hat{M}_{SS}^{(2)}\ddot{u}_{So}^{(2)} \\ &+ (C_{SS}^{(1)} + C_{SS}^{(2)})(\dot{u}_S^{(1)} + \dot{u}_{So}^{(2)}) - \hat{C}_{SS}^{(1)}\dot{u}_{So}^{(1)} - \hat{C}_{SS}^{(2)}\dot{u}_{So}^{(2)} \\ &+ (K_{SS}^{(1)} + K_{SS}^{(2)})(u_{So}^{(1)} u_{So}^{(2)}) - \hat{K}_{SS}^{(1)}u_{So}^{(1)} - \hat{K}_{SS}^{(2)}u_{So}^{(2)} \\ &- F_S + \hat{F}_S^{(1)} + \hat{F}_S^{(2)} \end{aligned} \tag{7}$$

Note that $r_S(t)$ can be obtained as a null vector, if the "caret" partitions in Eq. (3) are chosen to satisfy the following:

$$\begin{aligned} \hat{M}_{SS}^{(1)} &= \hat{M}_{SS}^{(2)} = M_{SS}^{(1)} + M_{SS}^{(2)} \\ \hat{C}_{SS}^{(1)} &= \hat{C}_{SS}^{(2)} = C_{SS}^{(1)} + C_{SS}^{(2)} \\ \hat{K}_{SS}^{(1)} &= \hat{K}_{SS}^{(2)} = K_{SS}^{(1)} + K_{SS}^{(2)} \\ \hat{F}_S^{(1)} &+ \hat{F}_S^{(2)} = F_S \end{aligned} \tag{8}$$

For a structure composed of more than two substructures, it is easily shown that the interface partition of the residual vector will be zero if the interface partitions of matrices in substructure problems are taken as the total corresponding partitions in the structure matrices, and if the interface partitions of the structure excitation are divided between substructures sharing the interface. Physically, this implies that substructures are modeled for independent computations as if all degrees of freedom in the structure beyond substructure interfaces are constrained to have zero response, and as if the excitation applied to the structure at interfaces is divided between substructures. Because of the simplification of the residual vector that results from these choices of the interface partitions, it will be arbitrarily assumed in this paper that these choices are adopted.

By superposition, the solution of the original response problem for the structure is equal to the sum of two responses, where the first is obtained by assembling solutions of the independent substructure response problems of Eq. (3) and summing at interfaces, and the second is the response of the structure to the negative of the residual defined in Eq. (5), which can be treated as an excitation. The response of the entire structure to the negative of the residual can be approximated with independent substructure-level calculations in the same way that the response of the structure to the original excitation was approximated. This results in a first correction $\Delta_1 u^{(k)}(t)$ for each substructure, which satisfies the substructure equations of motion

$$\begin{bmatrix} M_{LL}^{(k)} & M_{LS}^{(k)} \\ M_{SL}^{(k)} & \hat{M}_{SS}^{(k)} \end{bmatrix} \begin{Bmatrix} \Delta_1 \ddot{u}_L^{(k)} \\ \Delta_1 \ddot{u}_S^{(k)} \end{Bmatrix} + \begin{bmatrix} C_{LL}^{(k)} & C_{LS}^{(k)} \\ C_{SL}^{(k)} & \hat{C}_{SS}^{(k)} \end{bmatrix} \begin{Bmatrix} \Delta_1 \dot{u}_L^{(k)} \\ \Delta_1 \dot{u}_S^{(k)} \end{Bmatrix} + \begin{bmatrix} K_{LL}^{(k)} & K_{LS}^{(k)} \\ K_{SL}^{(k)} & \hat{K}_{SS}^{(k)} \end{bmatrix} \begin{Bmatrix} \Delta_1 u_L^{(k)} \\ \Delta_1 u_S^{(k)} \end{Bmatrix} = \begin{Bmatrix} -r_L^{(k)}(t) \\ \mathbf{0} \end{Bmatrix} \tag{9}$$

where $r_L^{(k)}(t)$ is given in Eq. (6). A refined approximation of the response will be in the form $u_o + \Delta_1 u$. If the structure response is represented in terms of these defined substructure responses, again with responses added at interfaces, the residual in the structure equations of motion will again be nonzero, because the structure response to the negative of the original residual was only approximated. Further refinements of substructure responses can be made, and each refinement will be an iteration toward reconciling the independent substructure responses that were originally computed. Numerical experiments implementing this approach show that the number of iterations required for achieving a given level of accuracy depends on the time step used in in integrating the response, as might be expected: Fewer iterations are required to reconcile the independently computed substructure responses when a smaller time step is used.

In the next section the iterative approach presented earlier is replaced with a direct method for reconciling independently computed substructure responses.

Efficient Reconciliation of Substructure Responses

If the response of the structure is to be calculated for more than a very short period of time using the iterative approach of the preceding section, it becomes worthwhile to streamline the reconciliation procedure. In this section a more efficient method for implementing the iterative reconciliation procedure is developed. Then a method for reconciling the independently calculated substructure responses directly, rather than iteratively, is presented.

In each iteration of the method of the previous section, the response of each substructure to the negative of a portion of the residual from the structure equations of motion is computed. From Eqs. (6) and (9) the portion of the residual appearing in the response problem for a substructure correction is always given in terms of only the interface motion computed for adjacent substructures. It is convenient, therefore, to introduce a vector $v^{(k)}$ that contains interface accelerations, velocities, and displacements computed for the kth substructure, i.e., $v^{(k)} = [\ddot{u}_S^{(k)T} \dot{u}_S^{(k)T} u_S^{(k)T}]^T$. Similarly, for a two-substructure case, a vector containing these interface response quantities computed for the other substructure can be denoted by $\bar{v}^{(k)}$. Then the substructure response problem for the first correction has the form

$$M^{(k)}\Delta_1 \ddot{u}^{(k)} + C^{(k)}\Delta_1 \dot{u}^{(k)} + K^{(k)}\Delta_1 u^{(k)} = \begin{bmatrix} 0 & 0 & 0 \\ -M_{LS}^{(k)} & -C_{LS}^{(k)} & -K_{LS}^{(k)} \end{bmatrix} \bar{v}_o^{(k)}(t) = f_o^{(k)}(t) \tag{10}$$

where the vector $f_o^{(k)}(t)$ is introduced to represent the interaction "force" exciting the kth substructure, due to the interface motion calculated originally for the adjacent substructure. This interaction force is not to be interpreted as an actual physical force exciting a physically uncoupled substructure, but rather an excitation of the discrete model of one substructure that accounts for interaction between substructures when interface motion for an adjacent substructure is prescribed. The representation of the interaction force will depend in general on how structure modeling at the interfaces is done. Because the correction in interface motion for this substructure, denoted by $\Delta_1 v^{(k)}$, is all that must be obtained so that the next correction for the other substructure can be made, it is not necessary to correct the response in local degrees of freedom. What is needed for streamlining the reconciliation process, then, is an efficient method for finding $\Delta_1 v^{(k)}(t)$ given $\bar{v}_o^{(k)}(t)$.

Ordinarily, $\bar{v}_o^{(k)}(t)$ will have been obtained for discrete times spaced Δt apart, where Δt is the time step used in integrating the substructure response. For simplicity the interaction force $f_o^{(k)}(t)$ can be approximated as being piecewise linear in time, although a more sophisticated approach can certainly be used.[13] It is assumed for convenience in this section that the response is to be calculated starting at time $t = 0$, and that the structure's initial conditions are such that interaction forces are zero at $t = 0$. In the next section these restrictions on the initial conditions will be removed. If substructure response has been calculated independently for $t > 0$, the correction in the interface response $\Delta_1 v^{(k)}(t)$ due to $\bar{v}_o^{(k)}(t)$, where $t > 0$, must be found. For the correction $\Delta_1 v^{(k)}(t)$ due to the adjacent substructure's interface motion only at the time $t = \Delta t$, that is, $\bar{v}_o^{(k)}(\Delta t)$, the piecewise linear representation of the interaction force takes the form

$$f_o^{(k)}(t) = \begin{cases} f_o^{(k)}(\Delta t)(t/\Delta t) & \text{for} \quad 0 \le t \le \Delta t \\ f_o^{(k)}(\Delta t)[2 - (t/\Delta t)] & \text{for} \quad \Delta t \le t \le 2\Delta t \\ \mathbf{0} & \text{for} \quad t \ge 2\Delta t \end{cases} \tag{11}$$

where $f_o^{(k)}(\Delta t)$ is defined as in Eq. (10). The first correction at time Δt is linearly related to both $f_o^{(k)}(\Delta t)$ and $\bar{v}_o^{(k)}(\Delta t)$ and can be written in the form

$$\Delta_1 v^{(k)}(\Delta t) \equiv \begin{Bmatrix} \Delta_1 \ddot{u}_S^{(k)}(\Delta t) \\ \Delta_1 \dot{u}_S^{(k)}(\Delta t) \\ \Delta_1 u_S^{(k)}(\Delta t) \end{Bmatrix} = \begin{bmatrix} A_S^{(k)}(\Delta t) \\ V_S^{(k)}(\Delta t) \\ D_S^{(k)}(\Delta t) \end{bmatrix} \bar{v}_o^{(k)}(\Delta t) = T_S^{(k)}(\Delta t)\bar{v}_o^{(k)}(\Delta t) \tag{12}$$

where the jth column of the submatrix $A_S^{(k)}(\Delta t)$, for example, contains the acceleration correction $\Delta_1 \ddot{u}_S^{(k)}(\Delta t)$ resulting from $\bar{v}_o^{(k)}(\Delta t) = e_j$, where e_j is a unit vector with a unit value in the jth entry. $V_S^{(k)}(\Delta t)$ and $D_S^{(k)}(\Delta t)$ are defined similarly. Equation (12) defines the matrix $T_S^{(k)}(\Delta t)$, which may be verbally defined as the matrix by which the adjacent substructure's motion in shared degrees of freedom at time $t = \Delta t$ must be multiplied to obtain a correction in the interface motion for the kth substructure at time $t = \Delta t$. It is convenient at this point to generalize this notation for use in the next section of the paper, by writing the first correction of interface motion at a later time $t = l\Delta t$, due to the interaction force of Eq. (11), which is in terms of the adjacent substructure's computed interface motion at the time $t = \Delta t$, as

$$\Delta_1 v^{(k)}(l\Delta t) = T_S^{(k)}(l\Delta t)\bar{v}_o^{(k)}(\Delta t) \tag{13}$$

This defines the matrix $T_S^{(k)}(l\Delta t)$.

As mentioned in the preceding section, the residual in the structure equations of motion is still nonzero after the first correction given by Eq. (12), because the correction is done independently for the different substructures. It is easily verified that the residual after the first correction, in the structure equations associated with the kth substructure, is in terms of $\Delta_1\bar{v}^{(k)}(\Delta t)$, rather than $\bar{v}_o^{(k)}(\Delta t)$. A second correction is therefore represented by the equation

$$\Delta_2 v^{(k)}(\Delta t) = T_S^{(k)}(\Delta t)\Delta_1\bar{v}^{(k)}(\Delta t) \tag{14}$$

where $\Delta_1\bar{v}^{(k)}(\Delta t)$ results from the first correction. More generally, the pth correction is given by

$$\Delta_p v^{(k)}(\Delta t) = T_S^{(k)}(\Delta t)\Delta_{p-1}\bar{v}^{(k)}(\Delta t) \tag{15}$$

When fully reconciled, the interface motion from the computed response of the kth substructure is given by the series

$$v_r^{(k)}(\Delta t) = v_o^{(k)}(\Delta t) + \Delta_1 v^{(k)}(\Delta t) + \Delta_2 v^{(k)}(\Delta t) + \cdots \tag{16}$$

where the subscripts r and o identify the fully reconciled and original independently calculated interface responses for the kth substructure, respectively. It should be noted that the final interface response for the structure is still the sum of components from different substructures.

In the case of a structure partitioned into two substructures, the final response for the structure in interface degrees of freedom becomes

$$\begin{aligned} v_r(\Delta t) &= v_r^{(1)}(\Delta t) + v_r^{(2)}(\Delta t) \\ &= [v_o^{(1)}(\Delta t) + T_S^{(1)}(\Delta t)\bar{v}_o^{(1)}(\Delta t) + \cdots] \\ &\quad + [v_o^{(2)}(\Delta t) + T_S^{(2)}(\Delta t)\bar{v}_o^{(2)}(\Delta t) + \cdots] \end{aligned} \tag{17}$$

In this case, $\bar{v}^{(1)} \equiv v^{(2)}$ and $\bar{v}^{(2)} \equiv v^{(1)}$; thus, the first correction term in $v_r^{(1)}(\Delta t)$ is $\Delta_1 v^{(1)}(\Delta t) = T_S^{(1)}(\Delta t)v_o^{(2)}(\Delta t)$. Because the second correction term in $v_r^{(1)}(\Delta t)$ is a result of the first correction term in $v_r^{(2)}(\Delta t)$, which is $\Delta_1 v^{(2)}(\Delta t) = T_S^{(2)}(\Delta t)v_o^{(1)}(\Delta t)$, it is given by

$$\Delta_2 v^{(1)}(\Delta t) = T_S^{(1)}(\Delta t)T_S^{(2)}(\Delta t)v_o^{(1)}(\Delta t) \tag{18}$$

Hence, Eq. (17) becomes, when more terms are retained,

$$\begin{aligned} v_r(\Delta t) = {} & [v_o^{(1)}(\Delta t) + T_S^{(1)}(\Delta t)v_o^{(2)}(\Delta t) \\ & + T_S^{(1)}(\Delta t)T_S^{(2)}(\Delta t)v_o^{(1)}(\Delta t) \\ & + T_S^{(1)}(\Delta t)T_S^{(2)}(\Delta t)T_S^{(1)}(\Delta t)v_o^{(2)}(\Delta t) + \cdots] \\ & + [v_o^{(2)}(\Delta t) + T_S^{(2)}(\Delta t)v_o^{(1)}(\Delta t) \\ & + T_S^{(2)}(\Delta t)T_S^{(1)}(\Delta t)v_o^{(2)}(\Delta t) \\ & + T_S^{(2)}(\Delta t)T_S^{(1)}(\Delta t)T_S^{(2)}(\Delta t)v_o^{(1)}(\Delta t) + \cdots] \end{aligned} \tag{19}$$

which can be rewritten as

$$\begin{aligned} v_r(\Delta t) = {} & \{I + T_S^{(1)}(\Delta t)T_S^{(2)}(\Delta t) + [T_S^{(1)}(\Delta t)T_S^{(2)}(\Delta t)]^2 \\ & + \cdots\} \times [v_o^{(1)}(\Delta t) + T_S^{(1)}(\Delta t)v_o^{(2)}(\Delta t)] \\ & + \{I + T_S^{(2)}(\Delta t)T_S^{(1)}(\Delta t) + [T_S^{(2)}(\Delta t)T_S^{(1)}(\Delta t)]^2 + \cdots\} \\ & \times [v_o^{(2)}(\Delta t) + T_S^{(2)}(\Delta t)v_o^{(1)}(\Delta t)] \end{aligned} \tag{20}$$

The matrix series in the preceding equation can be recognized as series representations for matrix inverses. This indicates that

$$\begin{aligned} v_r^{(1)}(\Delta t) = {} & [I - T_S^{(1)}(\Delta t)T_S^{(2)}(\Delta t)]^{-1}[v_o^{(1)}(\Delta t) \\ & + T_S^{(1)}(\Delta t)v_o^{(2)}(\Delta t)] \end{aligned} \tag{21}$$

and

$$\begin{aligned} v_r^{(2)}(\Delta t) = {} & [I - T_S^{(2)}(\Delta t)T_S^{(1)}(\Delta t)]^{-1}[v_o^{(2)}(\Delta t) \\ & + T_S^{(2)}(\Delta t)v_o^{(1)}(\Delta t)] \end{aligned} \tag{22}$$

and the actual structure response at the interface is their sum, as indicated in Eq. (17). Hence, the iterative approach to reconciliation can be replaced with a direct approach, and the fully reconciled interface motion for either substructure can be found independently given the originally computed interface motion for both substructures.

There is a more elegant way of arriving at Eqs. (21) and (22). After reconciliation of the responses of both substructures, the interface response of the first substructure must satisfy the equation

$$v_r^{(1)}(\Delta t) = v_o^{(1)}(\Delta t) + T_S^{(1)}(\Delta t)v_r^{(2)}(\Delta t) \tag{23}$$

where the first term on the right-hand side represents response to external excitation acting on the first substructure, and the second term is the response to the reconciled interface motion of the second substructure. Similarly, the interface response of the second substructure must satisfy the equation

$$v_r^{(2)}(\Delta t) = v_o^{(2)}(\Delta t) + T_S^{(2)}(\Delta t)v_r^{(1)}(\Delta t) \tag{24}$$

These two equations can be written together as

$$\begin{Bmatrix} v_r^{(1)}(\Delta t) \\ v_r^{(2)}(\Delta t) \end{Bmatrix} = \begin{Bmatrix} v_o^{(1)}(\Delta t) \\ v_o^{(2)}(\Delta t) \end{Bmatrix} + \begin{bmatrix} 0 & T_S^{(1)}(\Delta t) \\ T_S^{(2)}(\Delta t) & 0 \end{bmatrix} \begin{Bmatrix} v_r^{(1)}(\Delta t) \\ v_r^{(2)}(\Delta t) \end{Bmatrix} \tag{25}$$

Before reconciliation, the vectors $v_r^{(1)}(\Delta t)$ and $v_r^{(2)}(\Delta t)$ are unknowns, but this equation can be solved for them to yield

$$\begin{Bmatrix} v_r^{(1)}(\Delta t) \\ v_r^{(2)}(\Delta t) \end{Bmatrix} = \left\{ I - \begin{bmatrix} 0 & T_S^{(1)}(\Delta t) \\ T_S^{(2)}(\Delta t) & 0 \end{bmatrix} \right\}^{-1} \begin{Bmatrix} v_o^{(1)}(\Delta t) \\ v_o^{(2)}(\Delta t) \end{Bmatrix} \tag{26}$$

Equations (21) and (22) are obtained when the matrix inverse in Eq. (26) is partitioned appropriately.

This result is easily generalized for case involving more than two substructures. Letting a matrix T be defined as

$$T \equiv \begin{bmatrix} 0 & T_S^{(1)}(\Delta t) \\ T_S^{(2)}(\Delta t) & 0 \end{bmatrix} \tag{27}$$

for the two-substructure case, for a three-substructure case this matrix takes the form

$$T \equiv \begin{bmatrix} 0 & T_S^{(1,2)}(\Delta t) & T_S^{(1,3)}(\Delta t) \\ T_S^{(2,1)}(\Delta t) & 0 & T_S^{(2,3)}(\Delta t) \\ T_S^{(3,1)}(\Delta t) & T_S^{(3,2)}(\Delta t) & 0 \end{bmatrix} \tag{28}$$

where the submatrix $T_S^{(i,j)}(\Delta t)$ gives a correction in the ith substructure's interface degrees of freedom resulting from an interaction force in terms of the jth substructure's interface motion. Note that these submatrices are not fully populated unless all interface degrees of freedom are shared by all three substructures. The reconciled interface motion for the three substructures is then given by

$$\begin{Bmatrix} v_r^{(1)}(\Delta t) \\ v_r^{(2)}(\Delta t) \\ v_r^{(3)}(\Delta t) \end{Bmatrix} = [I - T]^{-1} \begin{Bmatrix} v_o^{(1)}(\Delta t) \\ v_o^{(2)}(\Delta t) \\ v_o^{(3)}(\Delta t) \end{Bmatrix} \tag{29}$$

Procedures for Occasional Reconciliation

The results of the preceding section provide an efficient direct method for reconciling the responses that are calculated independently for different substructures. However, the reconciliation must be done at each time step as it is presented there. It would be advantageous if this requirement were relaxed so that reconciliation could be done less frequently. This would allow the different processors to operate more independently, with communication between them occurring less frequently. It could also make it feasible to divide the structure model into more substructures than the number of available processors, because the cost of reading in new substructure data for processors would be reduced if it could be done less frequently. In this section the approach of the preceding section is extended so that response can be computed independently for substructures for an arbitrary number of time steps before reconciliation between substructures is done. At the end of a number of time steps over which substructure responses have been computed independently, reconciliation between substructures is done, with the result that the structure equation of motion is satisfied exactly at the end of every time step in this time period. The response in shared degrees of freedom is all that is involved in the reconciliation procedure, and at the end of a time period, the response in all degrees of freedom is updated. Hence, it is not necessary to compute the reconciled response in local degrees of freedom for every time step; they can simply be updated after the reconciliation is done for an entire time period, to give initial conditions for the next time period.

For brevity, procedures for reconciling substructure responses occasionally are presented for the two-substructure case in this section, since generalization to cases in which more substructures are involved is straightforward. After substructure response is computed independently for m time steps after the time $t = 0$, reconciliation must begin with the first time step $t = \Delta t$. This is done using the method of the preceding section, and the resulting reconciled response for the kth substructure in shared degrees of freedom can be written as the sum of two terms,

$$v_r^{(k)}(\Delta t) = v_o^{(k)} + T_S^{(k)}(\Delta t)\bar{v}_r^{(k)}(\Delta t) \tag{30}$$

although Eqs. (21) and (22) are actually used to obtain the reconciled responses. The first term in this equation represents the independently calculated response of the kth substructure, which is the response to the external excitation $F^{(k)}$ acting on the substructure. The second term is the response to the interaction force resulting from the adjacent substructure's reconciled motion in interface degrees of freedom. Note that, once $v_r^{(k)}(\Delta t)$ has been obtained using Eqs. (21) and (22), $\bar{v}_r^{(k)}(\Delta t)$ will be available as

$$\bar{v}_r^{(k)}(\Delta t) = \bar{v}_o^{(k)}(\Delta t) + \bar{T}_S^{(k)}(\Delta t)v_r^{(k)}(\Delta t) \tag{31}$$

where $\bar{T}_S^{(k)}(\Delta t)$ simply refers to the $T_S^{(k)}(\Delta t)$ matrix for the adjacent substructure.

For the time $t = 2\Delta t$, the kth substructure's interface motion before reconciliation, after being updated to reflect the reconciliation at the time $t = \Delta t$, can also be written as the sum of two terms:

$$v_u^{(k)}(2\Delta t) = v_o^{(k)}(2\Delta t) + T_S^{(k)}(2\Delta t)\bar{v}_r^{(k)}(\Delta t) \tag{32}$$

The two terms are analogous to those in Eq. (30), and the subscript u is used to indicate that this is an updated, but not yet reconciled, quantity. The matrix $T_S^{(k)}(2\Delta t)$ is defined in Eq. (13). The reconciled response of the kth substructure at the time $t = 2\Delta t$ in shared degrees of freedom is obtained in the same way as that in Eqs. (21) and (22) and is given by

$$v_r^{(k)}(2\Delta t) = [I - T_S^{(k)}(\Delta t)\bar{T}_S^{(k)}(\Delta t)]^{-1} \times [v_u^{(k)}(2\Delta t) + T_S^{(k)}(\Delta t)\bar{v}_u^{(k)}(2\Delta t)] \tag{33}$$

where $\bar{v}_u^{(k)}(2\Delta t)$ is obtained from

$$\bar{v}_u^{(k)}(2\Delta t) = \bar{v}_o^{(k)}(2\Delta t) + \bar{T}_S^{(k)}(2\Delta t)v_r^{(k)}(\Delta t) \tag{34}$$

Generalizing these results, if substructure responses have been reconciled for $l - 1$ time steps, the updated interface motion associated with the kth substructure at the time $t = l\Delta t$ becomes

$$v_u^{(k)}(l\Delta t) = v_o^{(k)}(l\Delta t) + \sum_{i=1}^{l-1} T_S^{(k)}[(l - i + 1)\Delta t]\bar{v}_r^{(k)}(i\Delta t) \tag{35}$$

and similarly,

$$\bar{v}_u^{(k)}(l\Delta t) = \bar{v}_o^{(k)}(l\Delta t) + \sum_{i=1}^{l-1} \bar{T}_S^{(k)}[(l - i + 1)\Delta t]v_r^{(k)}(i\Delta t) \tag{36}$$

The reconciled interface response for the kth substructure becomes

$$v_r^{(k)}(l\Delta t) = [I - T_S^{(k)}(\Delta t)\bar{T}_S^{(k)}(\Delta t)]^{-1}[v_u^{(k)}(l\Delta t) + T_S^{(k)}(\Delta t)\bar{v}_u^{(k)}(l\Delta t)] \tag{37}$$

and the vector $\bar{v}_r^{(k)}(l\Delta t)$ can be obtained using

$$\bar{v}_r^{(k)}(l\Delta t) = \bar{v}_u^{(k)}(l\Delta t) + \bar{T}_S^{(k)}(\Delta t)v_r^{(k)}(l\Delta t) \tag{38}$$

After reconciliation has been done for all m time steps, the displacement and velocity in all local degrees of freedom must be corrected to provide initial conditions for independent substructure response computation for another m time steps. This is done by again considering the substructure response to be the sum of response computed independently and response to the excitation due to interaction with the adjacent substructure. By analogy with Eq. (12), a correction in the local velocity and displacement at a time $t = l\Delta t$, resulting from adjacent substructures' interface motion at time $t = \Delta t$, can be written as

$$\begin{Bmatrix} \Delta\dot{u}_L^{(k)}(l\Delta t) \\ \Delta u_L^{(k)}(l\Delta t) \end{Bmatrix} = \begin{bmatrix} V_L^{(k)}(l\Delta t) \\ D_L^{(k)}(l\Delta t) \end{bmatrix} \bar{v}_r^{(k)}(\Delta t) = T_L^{(k)}(l\Delta t)\bar{v}_r^{(k)}(\Delta t) \tag{39}$$

Then the updated local velocity and displacement at time $t = m\Delta t$, taking all of the reconciliation into account, are given by

$$\begin{Bmatrix} \dot{u}_{Lu}^{(k)}(m\Delta t) \\ u_{Lu}^{(k)}(m\Delta t) \end{Bmatrix} = \begin{Bmatrix} \dot{u}_{Lo}^{(k)}(m\Delta t) \\ u_{Lo}^{(k)}(m\Delta t) \end{Bmatrix} + \sum_{i=1}^{m} T_L^{(k)}[(m - i + 1)\Delta t]\bar{v}_r^{(k)}(i\Delta t) \tag{40}$$

For the response after $t = m\Delta t$, the procedure that was followed for $t \leq m\Delta t$ could be followed, except for the fact that the interaction force can be nonzero at $t = m\Delta t$. Letting $\hat{t} = m\Delta t$ for generality, so that $\hat{t}$ can represent any time when substructures have been updated completely, in the piecewise linear representation of $f^{(k)}(t)$ a term will have to be added of the form

$$f^{(k)}(t) = \begin{cases} f^{(k)}(\hat{t})\left(1 - \dfrac{t - \hat{t}}{\Delta t}\right) & \text{for} \quad \hat{t} \leq t \leq \hat{t} + \Delta t \\ 0 & \text{for} \quad t \geq \hat{t} + \Delta t \end{cases} \tag{41}$$

This will necessitate the definition of the matrices $T_{SO}^{(k)}(l\Delta t)$ and $\bar{T}_{SO}^{(k)}(l\Delta t)$, $l = 1,2,\ldots,m$, and $T_{LO}^{(k)}(m\Delta t)$, which represent response at $t = \hat{t} + l\Delta t$ or $t = \hat{t} + m\Delta t$ to adjacent substructures' interface motion at $t = \hat{t}$, by the equations

$$\Delta v^{(k)}(\hat{t} + l\Delta t) = T_{SO}^{(k)}(l\Delta t)\bar{v}_r^{(k)}(\hat{t}) \tag{42}$$

and

$$\Delta \bar{v}^{(k)}(\hat{t} + l\Delta t) = \bar{T}_{SO}^{(k)}(l\Delta t)v_r^{(k)}(\hat{t}) \tag{43}$$

and

$$\begin{Bmatrix} \Delta \dot{u}_L^{(k)}(\hat{t} + m\Delta t) \\ \Delta u_L^{(k)}(\hat{t} + m\Delta t) \end{Bmatrix} = T_{LO}^{(k)}(m\Delta t)\bar{v}_r^{(k)}(\hat{t}) \tag{44}$$

Allowing for nonzero interaction force at $t = \hat{t}$ will require the vectors of interface motion for the kth substructure to be updated before reconciliation can be performed for the time $\hat{t} + \Delta t$. The updated vectors are given by

$$v_u^{(k)}(\hat{t} + \Delta t) = v_o^{(k)}(\hat{t} + \Delta t) + T_{SO}^{(k)}(\Delta t)\bar{v}_r^{(k)}(\hat{t}) \tag{45}$$

and

$$\bar{v}_u^{(k)}(\hat{t} + \Delta t) = \bar{v}_o^{(k)}(\hat{t} + \Delta t) + \bar{T}_{SO}^{(k)}(\Delta t)v_r^{(k)}(\hat{t}) \tag{46}$$

The reconciled interface responses at $t = \hat{t} + \Delta t$ are

$$v_r^{(k)}(\hat{t} + \Delta t) = [I - T_S^{(k)}(\Delta t)\bar{T}_S^{(k)}(\Delta t)]^{-1} \times [v_u^{(k)}(\hat{t} + \Delta t) + T_S^{(k)}(\Delta t)\bar{v}_u^{(k)}(\hat{t} + \Delta t)] \tag{47}$$

and

$$\bar{v}_r^{(k)}(\hat{t} + \Delta t) = \bar{v}_u^{(k)}(\hat{t} + \Delta t) + \bar{T}_S^{(k)}(\Delta t)v_r^{(k)}(\hat{t} + \Delta t) \tag{48}$$

Updated interface responses at a time $t = \hat{t} + l\Delta t$ take the form

$$v_u^{(k)}(\hat{t} + l\Delta t) = v_o^{(k)}(\hat{t} + l\Delta t) + T_{SO}^{(k)}(l\Delta t)\bar{v}_r^{(k)}(\hat{t}) + \sum_{i=1}^{l-1} T_S^{(k)}[(l - i + 1)\Delta t]\bar{v}_r^{(k)}(\hat{t} + i\Delta t) \tag{49}$$

and

$$\bar{v}_u^{(k)}(\hat{t} + l\Delta t) = \bar{v}_o^{(k)}(\hat{t} + l\Delta t) + \bar{T}_{SO}^{(k)}(l\Delta t)v_r^{(k)}(\hat{t}) + \sum_{i=1}^{l-1} \bar{T}_S^{(k)}[(l - i + 1)\Delta t]v_r^{(k)}(\hat{t} + i\Delta t)] \tag{50}$$

and the reconciled interface responses at the same time are given by

$$v_r^{(k)}(\hat{t} + l\Delta t) = [I - T_S^{(k)}(\Delta t)\bar{T}_S^{(k)}(\Delta t)]^{-1} \times [v_u^{(k)}(\hat{t} + l\Delta t) + T_S^{(k)}(\Delta t)\bar{v}_u^{(k)}(\hat{t} + l\Delta t)] \tag{51}$$

and

$$\bar{v}_r^{(k)}(\hat{t} + l\Delta t) = \bar{v}_u^{(k)}(\hat{t} + l\Delta t) + \bar{T}_S^{(k)}(\Delta t)v_r^{(k)}(\hat{t} + l\Delta t) \tag{52}$$

Finally, the updated local velocity and displacement at time $t = \hat{t} + m\Delta t$ are given by

$$\begin{Bmatrix} \dot{u}_{Lu}^{(k)}(\hat{t} + m\Delta t) \\ u_{Lu}^{(k)}(\hat{t} + m\Delta t) \end{Bmatrix} = \begin{Bmatrix} \dot{u}_{Lo}^{(k)}(\hat{t} + m\Delta t) \\ u_{Lo}^{(k)}(\hat{t} + m\Delta t) \end{Bmatrix} + T_{LO}^{(k)}(m\Delta t)\bar{v}_r^{(k)}(\hat{t}) + \sum_{i=1}^{m} T_L^{(k)}[(m - i + 1)\Delta t]\bar{v}_r^{(k)}(\hat{t} + i\Delta t) \tag{53}$$

With these developments, nonzero velocity and displacement at $t = 0$ can now be handled, by letting $\hat{t} = 0$ and dividing the initial interface velocity and displacement arbitrarily between the substructures sharing the interface, and using Eqs. (45-53).

The preparatory computation that must be done for implementation of this algorithm consists of generating the matrices $T_{SO}^{(k)}(l\Delta t)$, $T_S^{(k)}(l\Delta t)$, and $T_L^{(k)}(l\Delta t)$, for $l = 1, \ldots, m$, and the matrix $T_{LO}^{(k)}(m\Delta t)$, for each substructure. These matrices are obtained by solving response problems as described in the text following Eq. (12).

Numerical Example

The algorithm of this paper is demonstrated on an example problem involving the structure shown in Fig. 1. The structure

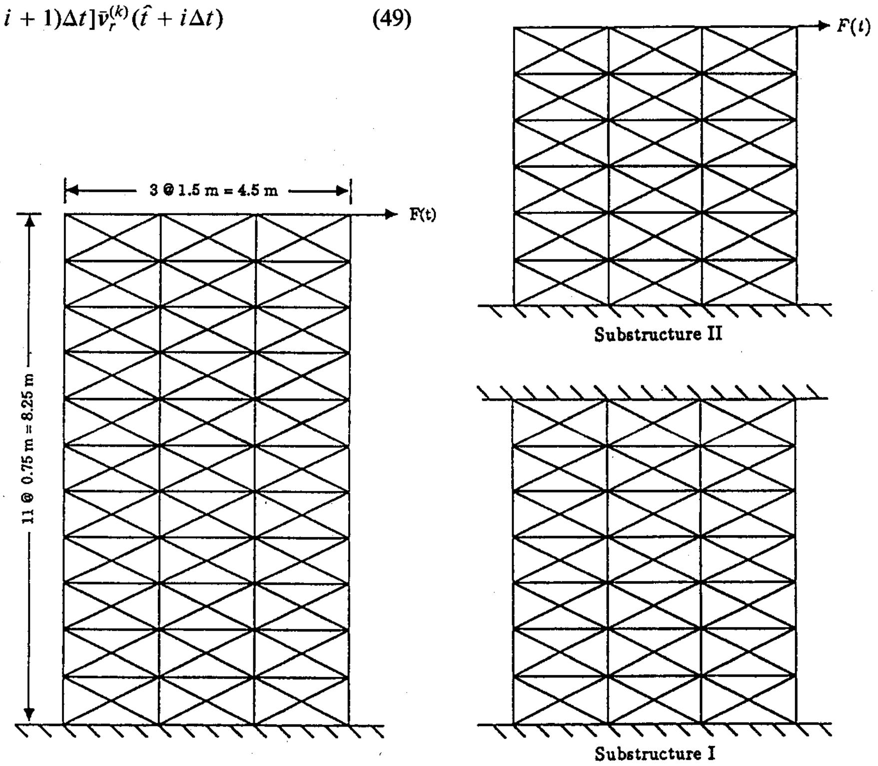

Fig. 1 Plane truss used in the numerical example and its division into substructures.

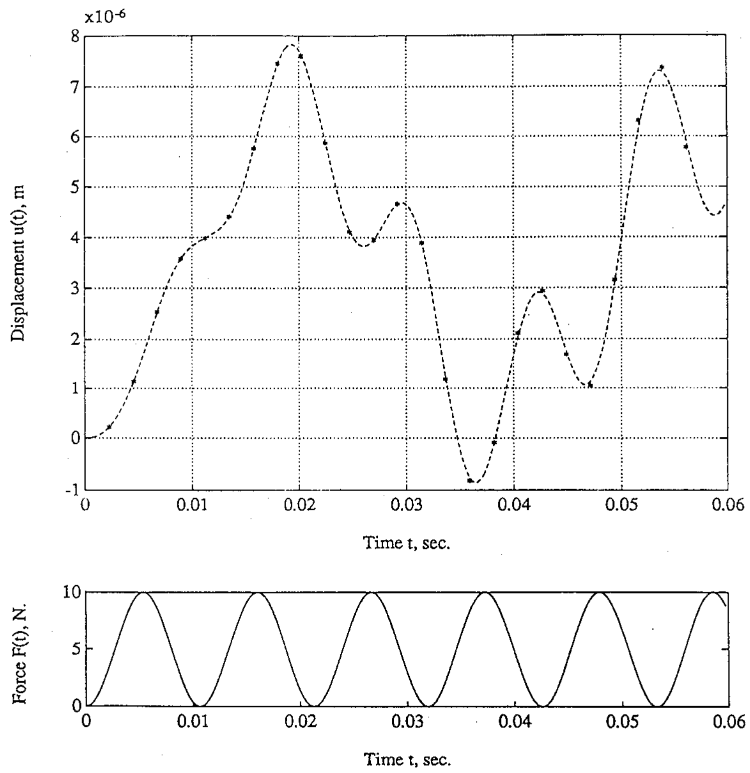

Fig. 2 Plots of exact response (dashed line) and computed response (asterisks) and excitation.

is a plane truss composed of 143 aluminum members, each of which has an elastic modulus of $E = 70 \times 10^9$ N/m^2, a cross-sectional area of $A = 4 \times 10^{-4}$m^2, and a density of $\rho = 2710$ kg/m^3. The dimensions are as shown. A force in N is applied to the top right corner of the truss and is given by

$$F(t) = 5(1 - \cos\Omega t)$$

where $\Omega = 590.3$ rad/s, which is between the second and third natural frequencies of the structure. Linear finite elements are used to represent truss members. There are a total of 88 degrees of freedom, and the structure is assumed to have proportional damping, i.e., $C = \alpha M + \beta K$, where α and β were chosen to give modal damping factors between 1 and 5%. For implementation of the algorithm the structure was divided at the top of the sixth bay into two substructures, which are also shown in Fig. 1. Note that, for Eq. (7) to be satisfied, each substructure is modeled for independent computation as if it were clamped one truss bay beyond the interface, as shown in the figure.

In Fig. 2 the response of the structure at the point of application of the force is plotted above a plot of excitation. The dashed line represents the exact response, and the asterisks represent values that were obtained using the algorithm of this paper. The responses of the two substructures were obtained using an algorithm that finds the exact response to a piecewise linear approximation of the excitation,[14] and a time step of $\Delta t = 3.74 \times 10^{-4}$ s was used, which is equal to about 1/28 of the period of the excitation, and is also approximately equal to the period of the highest mode of the structure. The substructure responses were computed independently for six time steps at a time and then reconciled in interface degrees of freedom for every time step and updated in local degrees of freedom at every sixth time step. Because the force was applied at a local degree of freedom for one of the substructures, the reconciled response of that degree of freedom was only computed at every sixth time step, when local degrees of freedom were updated. From the plot of Fig. 2, it is evident that the accuracy obtained is quite adequate, even for the relatively large time step used. Note that the only approximations involved in obtaining these results are in the piecewise linear representations of the excitation and the interaction forces between substructures.

Conclusions

In this paper an algorithm is presented for the parallel computation of transient response of structures, where the parallelization is based on a division of the structure into substructures. Response is computed independently for the different substructures for an arbitrary number of time steps, and then the motion at interfaces is reconciled for each time step using an efficient direct procedure. At the end of the time period over which substructure responses are calculated independently, response in all substructure degrees of freedom is updated, so that independent computation of substructure response can proceed again. The fact that reconciliation can be done occasionally, rather than at every time step, means that the different processors can operate more independently than has been possible with previous algorithms. Also, because the reconciliation process requires very little effort, the total amount of computation required is not much greater than that required for the independent substructure response calculations, so that parallel processors can be put to efficient use. The algorithm as it is presented here uses a linear approximation of the interaction force between substructures over every time step. The results are very accurate, which is to be expected because the computed response exactly satisfies the structure equations of motion at the end of each time step.

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